Problem Set 2 - LV 141.A55 QISS - 14.3.2016

1. **Qubit States**

   (a) Which of the following states are valid qubit states

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i\,|1\rangle\right)$$
$$|\Psi\rangle = \frac{1}{4}\,|0\rangle + \frac{3}{4}\,|1\rangle$$
$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(i\,|0\rangle + |1\rangle\right)$$
$$|\Psi\rangle = -0.6\,|0\rangle + 0.8\,|1\rangle$$
$$|\Psi\rangle = \sqrt{\frac{1}{2}}\sqrt{1+\frac{1}{\sqrt{3}}}\,|0\rangle + \frac{1+i}{2}\sqrt{1-\frac{1}{\sqrt{3}}}\,|1\rangle$$

   For the valid qubit states, calculate θ and φ, i.e. $|\Psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\varphi}\sin(\theta/2)\,|1\rangle$

   (b) Which of the matrices describe valid qubit states

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} \frac{3}{4} & \frac{-i}{4} \\ \frac{i}{4} & \frac{1}{4} \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2} & \frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} \frac{3+\sqrt{3}}{6} & \frac{1+i}{2\sqrt{3}} \\ \frac{1-i}{2\sqrt{3}} & \frac{3-\sqrt{3}}{6} \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2} & \frac{-1+i}{2\sqrt{2}} \\ \frac{-1-i}{2\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} 0.9607 & 0.1661 - 0.1006i \\ 0.1661 + 0.1006i & 0.0393 \end{pmatrix}$$

   Are the states pure or mixtures? Calculate $s_x = \langle\sigma_x\rangle$, $s_y = \langle\sigma_y\rangle$, $s_z = \langle\sigma_z\rangle$

2. **Qubit Hamiltonian**

   Consider the following general qubit Hamiltonian

$$H = A\sigma_z + D\sigma_x = \begin{pmatrix} A & D \\ D & -A \end{pmatrix}$$

   Calculate the eigenstates and eigenenergies of this Hamiltonian.

   What is the transformation from the initial states $|0\rangle, |1\rangle$ to the eigenstates of the Hamiltonian? Interpret your result as a rotation in the Bloch sphere.

3. Rabi's Formula In the lecture we discussed the dynamics of a qubit in the presence of an oscillatory driving field.

$$H = \frac{\hbar\omega_0}{2}\sigma_z + \frac{A}{2}\left(\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t)\right)$$

(a) Show that in a rotating frame the Schrödinger equation has the following form

$$i\frac{d}{dt}\begin{pmatrix} b_1(t) \\ b_0(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix}\begin{pmatrix} b_1(t) \\ b_0(t) \end{pmatrix}$$

where $\Delta\omega = \omega_0 - \omega$ and $\hbar\omega_1 = A$

(b) Calculate the probability of being in the excited state $|1\rangle$ under the assumption that at time zero $t = 0$ the system is in the ground state $|\Psi(0)\rangle = |0\rangle$

$$P_1(t) = |\langle 1|\Psi(t)\rangle|^2 = \left|\left\langle 1|\tilde{\Psi}(t)\right\rangle\right|^2$$

(c) Calculate the time average of the probability of being in the excited state

$$\bar{P}_1 = \lim_{t\to\infty} \frac{1}{T}\int_0^T P_1(t)dt$$

(d) Plot the Rabi frequency (frequency of the oscillation of $P_1(t)$) as a function of the field amplitude A.